ORTHOGONAL FUNCTIONS

Overview

Throughout the first half of the semester we have become familiar with the concept of orthogonal coordinate systems. A coordinate system is orthogonal if its base vectors satisfy the relationship:

\[ \hat{e}_i \cdot \hat{e}_j = \delta_{ij} \]  

(1)

Where \( \hat{e} \) represents a base vector, and \( \delta_{ij} \) is the Kronecker delta. The two base vectors have a dot product of zero if they are different, and a dot product of one if they are the same.

We saw briefly that the concept of orthogonality can be applied to functions (see Question #5 on Homework #2). We will see in Chapters 7, 8 and 13 of Boas that understanding orthogonality of functions gives us a powerful tool in solving a wide array of mathematical physics problems. (Griffiths calls this the "Fourier trick" in Chapter 3 of his book.)

Over the course of the next few weeks, we will learn about many types of orthogonal sets of functions, but to begin will examine the behavior of our old friends, sin and cos.

Periodic Behavior of Trig Functions

Let's start by plotting the functions \( \sin x \), \( \sin 2x \) and \( \sin 3x \) on the same graph:

\[ \text{In}[5]= \text{Plot}[[\sin[x], \sin[2x], \sin[3x]], \{x, 0, 2\pi\}] \]

You can see how to plot simple functions by reviewing the input line [1]. The basic command to plot a function is simply "Plot". The function is followed by square brackets. Since we want to plot three different functions on the same set of axes, we use braces (curly brackets) to indicate the list of functions to plot. Finally, we must indicate the range of the plot; this is done in the final set of braces which instructs Mathematica to plot these functions from \( x = 0 \) to \( x = 2\pi \). This plot illustrates well that the sin function is periodic with a periodicity of 2 \( \pi \). \( \sin 2x \) and \( \sin 3x \) have shorter periods, but they are still 2\( \pi \) periodic.

We observe the same periodicity for \( \cos (nx) \):
Orthogonality of Cos and Sin

Since \( \sin(nx) \) and \( \cos(nx) \) have periodicities of \( 2\pi \) if \( n \) is an integer, it is reasonable to study the behavior of these functions over the interval \([-\pi, \pi]\) (which is an interval of length \( 2\pi \)). Let's extend our understanding of the concept of orthogonality from vectors to functions by considering the integral:

\[
\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx
\]

(2)

where \( n \) and \( m \) are integers. First, let's recall how to do integrals using Mathematica. We can solve this integral using the following Mathematica command:

```mathematica
In[8]:= Integrate[Cos[n x], {x, -\pi, \pi}]
```

Output line [8] provides the answer. It is a simple matter to realize that this integral has a value of zero for integral values of \( n \), since the value of \( \sin(n\pi) \) is zero for all integers.

This is nice, but we can throw around a little Mathematica firepower and make our lives even easier. Consider the following line of input:

```mathematica
In[9]:= Integrate[Cos[n x], {x, -\pi, \pi}, Assumptions \rightarrow \text{Element}[n, \text{Integers}]]
```

Output line [9] shows how you can specify certain conditions. The arrow that follows "Assumptions" is made by striking the dash key followed by the ">" key without an intervening space. (Try typing this into your notebook to see what happens.) Input line [9] instructs the program to integrate \( \cos(nx) \) between the stated limits, but now with the assumption that \( n \) is a member of the domain of integers.

Now, that we have gotten some practice in doing integrals with Mathematica, let's investigate the orthogonality of these func-
tions. We remember from homework #2 that functions are orthogonal on an interval if the integral of their product is zero if the functions are different, and non zero if the functions are the same. In other words, we are interested in evaluating:
\[
\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx
\]  
(3)

Let's use Mathematica to solve this symbolically:

\[
\text{In[15]:= Integrate[Cos[m x] Cos[n x], \{x, -\pi, \pi\}]
\]

\[
\text{Out[15]= } \frac{2 m \cos[n \pi] \sin[m \pi] - 2 n \cos[m \pi] \sin[n \pi]}{m^2 - n^2}
\]

If you look at this seemingly messy expression, you will notice pretty quickly that if m and n are both integers, the numerator must go to zero because of the \(\sin(n \pi)\) and \(\sin(m \pi)\) terms in the numerator, as we have already determined that \(\sin(n\pi)\) is zero for all integers. So the integral in equation (3) is always zero, right? Well, almost.

Let's use our newly found Mathematica trick to see what we get if we set both m and n equal to integers:

\[
\text{In[16]:= Integrate[Cos[n x] Cos[m x], \{x, -\pi, \pi\}, Assumptions -> Element\{\{m, n\}, \text{Integers}\}]}
\]

\[
\text{Out[16]= 0}
\]

Notice that we use braces since we have a set of two integers. Happily we obtain the result zero. We can illustrate another useful feature of Mathematica by doing the same calculation in a slightly different format. In this case, we will assume that m is an integer and then separately assume n is an integer:

\[
\text{In[17]:= Integrate[Cos[n x] Cos[m x], \{x, -\pi, \pi\}, Assumptions -> Element[m, \text{Integers}] \&\& Element[n, \text{Integers}] ]}
\]

\[
\text{Out[17]= 0}
\]

Here, we use the "\&\&" to represent the "and" function; meaning both statements must simultaneously be true. Again, we find that the integral in eq. (2) is zero if m and n are different integers.

But what is the result when \(m = n\). We already determined that the numerator is zero if both m and n are integers. However, if you look at the result in output line[15], you should notice that the denominator also goes to zero if m equals n.

Thus, in the case where m and n are equal integers, we have an indeterminate form (remember L'Hôpital's Rule?). So we have to consider separately the case of m=n. We could do this the easy way by direct computation:

\[
\text{In[19]:= Integrate[Cos[n x] Cos[n x], \{x, -\pi, \pi\}, Assumptions -> Element[n, \text{Integers}] ]}
\]

\[
\text{Out[19]= } \pi
\]

Or do it a slightly more sophisticated way:

\[
\text{In[18]:= Integrate[Cos[n x] Cos[m x], \{x, -\pi, \pi\}, Assumptions -> Element[n, \text{Integers}] \&\& m \equiv n]}
\]

\[
\text{Out[18]= } \pi
\]

Here we explicitly note the two conditions that n is an integer and that \(m = n\). You might notice that we used a "double equal" sign here; Mathematica has three different types of equal signs, but those details are for later.

For now, the important point is that we have shown that \(\cos(nx)\) is in fact an orthogonal function over the interval \([-\pi, \pi]\). Our
original integral in eq. (3) is zero if m and n are different, and equals \( \pi \) if m and n are the same.

This means that \( \cos(nx) \) defines a set of orthogonal functions on the interval \([-\pi,\pi]\).

Not surprisingly, we will find that \( \sin(nx) \) is also orthogonal over the interval \([-\pi,\pi]\).

We evaluate:

\[
\text{In}[20]:= \text{Integrate}[\sin(mx) \sin(mx), \{x, -\pi, \pi\}, \text{Assumptions} \rightarrow \text{Element}\{m, n\}, \text{Integers}] \\
\text{Out}[20]= 0
\]

and

\[
\text{In}[21]:= \text{Integrate}[\sin(nx) \sin(mx), \{x, -\pi, \pi\}, \text{Assumptions} \rightarrow \text{Element}\{m, n\}, \text{Integers} \&\& m = n] \\
\text{Out}[21]= \pi
\]

And we confirm that \( \sin (nx) \) is orthogonal on the interval \([-\pi,\pi]\).

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**Exploiting Orthogonality/Fourier Series**

We have spent considerable time and learned several new Mathematica tricks to show that \( \sin (nx) \) and \( \cos (nx) \) represent sets of orthogonal functions on the interval \([-\pi,\pi]\). So why is all of this important?

You learned in Calc II that you can expand nicely behaved functions in terms of an infinite series of polynomials (known as Taylor series). Some of the most famous of these are:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}
\]

(4)

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\]

(5)

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots
\]

(6)

In general terms, you can expand many well behaved functions as a Taylor series around the point \( x = 0 \) as:

\[
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n
\]

(7)

If you consider the general form of the series expansion shown above, you can think of Taylor series as all being the same, in other words, all Taylor series look exactly like eq. (7). The only catch is that you have to figure out the coefficients.

There are many other types of series expansions you will learn in physics courses, and this semester we will begin introducing you to a number of them. In each case, keep in mind that there is a common form for all series of a certain category ... the trick is figuring out the coefficients, and that's where orthogonality comes in.

Chapter 7 of Boas introduces Fourier series. Fourier series have wide applicability to many physical problems, some of which we will see at the end of the semester. Like Taylor series, Fourier series allow us to express a function as an infinite sum of terms. In particular, we can take almost any appropriately well behaved function (the Dirichlet conditions that we will study in section 6
quantify what constitutes a sufficiently well behaved function) and express it in terms of an infinite set of sin and cos terms. In other words, we can take just about any \( f(x) \) and write it as:

\[
f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ... \\
+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ...
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]  \hspace{1cm} (8)

And there you have it. That’s how you write any function as a Fourier series. The only detail we have omitted is that we need to figure out how to compute the coefficients.